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Quantum Fluctuations of the Current and Voltage in Thermal Vacuum State for Mesoscopic Quartz Piezoelectric Crystal

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The mesoscopic quartz piezoelectric crystal equivalent circuit is quantized by the method of damped harmonic oscillator quantization. It is shown that, when each branch is in the thermal vacuum states, the quantum fluctuations of the voltage, and current of each loop relate with not only the equivalent circuit inherent parameter, but also the temperature and decay according to exponent along with time.

KEY WORDS: mesoscopic quartz piezoelectric crystal; equivalent circuit; thermal vacuum state; quantum fluctuation.

1. INTRODUCTION

The quartz piezoelectric crystal named the quartz harmonic oscillator has been applied to filter and oscillation, etc. With the rapid development of nanoelectronics, the trend toward miniaturization of circuits and components is obvious more and more. Quantum effects in electronic devices and circuits should be taken into account when the transport dimension reaches a characteristic dimension.

Recently, the quantum effects of circuit and device have become one of the hotspot in mesoscopic physics. A lot of literatures (Chen *et al.*, 1995; Fan *et al.*, 2000, 2002; Ji and Lei, 2001; Li, 2005; Li *et al.*, 1996; Liang and Yuan, 2002; Liu *et al.*, 2003; Song, 2003a,b; Wang, 2002; Wang *et al.*, 2000a,b; Zhang *et al.*, 2002) have studied quantum fluctuation of the electric charge, voltage and current in LC, RLC, capacitance coupling, inductance coupling, LC parallel connection, RLC series-parallel connection and RLC parallel connection circuits. In this paper, we shall study quantum fluctuations of the current and voltage in thermal vacuum state

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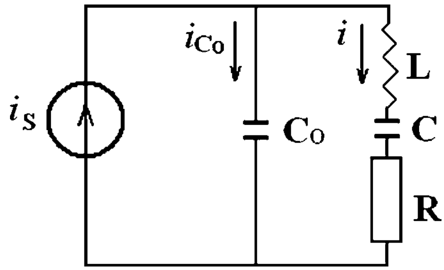

Fig. 1. Quartz piezoelectric crystal equivalent circuit.

for mesoscopic quartz piezoelectric crystal by the method of damped harmonic oscillator quantization (Peng, 1980).

2. THE QUANTIZATION OF THE MESOSCOPIC QUARTZ PIEZOELECTRIC CRYSTAL EQUIVALENT CIRCUIT

Figure 1 stands for the quartz piezoelectric crystal equivalent circuit. It can be regarded as parallel-board capacitor when crystal is not oscillation, the C_0 standing for static capacitance relating with wafer dimension and electrode area.

When crystal is oscillation, the equivalent inductance L stands for mechanism oscillation inertia, the equivalent capacitance C stands for wafer spring. The numerical values of the L and C relating with wafer incision, wafer electrode dimension and shape, the equivalent resistance R stands for friction ullage. According to Kirchhoff's Law, the classical equation of motion of the system is

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \left(\frac{1}{C_0} + \frac{1}{C}\right)i = \frac{i_S(t)}{C_0}, \tag{1}$$

where i is the electric current of transflux inductance loop, $i_s(t)$ is electric current of message source. Defining $u = L\frac{di}{dt}$, we have

$$\dot{i} = \frac{u}{L}, \quad \dot{u} = -\frac{R}{L}u - \left(\frac{1}{C_0} + \frac{1}{C}\right)i + \frac{i_S(t)}{C_0}. \tag{2}$$

From Eq. (2), we obtain

$$\frac{\partial \dot{i}}{\partial i} + \frac{\partial \dot{u}}{\partial u} = -\frac{R}{L}, \frac{d}{dt}[i, u] = -\frac{R}{L}[i, u]. \tag{3}$$

Obviously, when $R \neq 0$, the i and u are not conjugate variable in classical conditions, the common quantum condition must be modified to satisfy the following damped commutation relation

$$[i, u] = j\hbar\tau^{-2}\mathrm{e}^{-\frac{R}{L}t}, \quad (j^2 = -1) \tag{4}$$

where τ stands for the unit time constant. We consider the following transformations

$$i = I\tau^{-1}e^{-\frac{R}{2L}t}, \quad u = \tau^{-1}\left(U - \frac{R}{2}I\right)e^{-\frac{R}{2L}t}. \tag{5}$$

One has

$$[I, U] = j\hbar, \tag{6}$$

where the I and U stand for plural canonical current and plural canonical voltage, respectively. From Eq. (5), we obtain

$$\dot{I} = \frac{U}{L}, \quad \dot{U} = -L\omega^2 I + \frac{\tau}{C_0}e^{\frac{R}{2L}t}i_S(t), \tag{7}$$

where

$$\omega^2 = \omega_0^2 - \frac{R^2}{4L^2}, \quad \omega_0^2 = \frac{1}{L}\left(\frac{1}{C_0} + \frac{1}{C}\right).$$

According to Eq. (7) and canonical Hamiltonian equation

$$\dot{I} = \frac{\partial H}{\partial U}, \quad \dot{U} = -\frac{\partial H}{\partial I}, \tag{8}$$

we obtain

$$H = \frac{U^2}{2L} + \frac{1}{2}L\omega^2 I^2 - \frac{\tau}{C_0}e^{\frac{R}{2L}t}i_S(t)I. \tag{9}$$

Introducing

$$A = \frac{1}{\sqrt{2L\omega\hbar}}(L\omega I + jU), \quad A^+ = \frac{1}{\sqrt{2L\omega\hbar}}(L\omega I - jU). \tag{10}$$

We can prove that

$$[A, A^+] = 1 \tag{11}$$

The Hamiltonian of this system is

$$H = \hbar\omega\left(A^+A + \frac{1}{2}\right) - (A + A^+)\sqrt{\frac{\hbar}{2L\omega}}\frac{\tau}{C_0}e^{\frac{R}{2L}t}i_S(t), \tag{12}$$

which shows that when the mesoscopic quartz piezoelectric crystal equivalent circuit is quantized, it is equivalent to quantized harmonic oscillator under the electrical source.

3. THERMAL VACUUM STATE

We introduce a relative tilde space besides the Hilbert space in TFD theory (Umezawa *et al.*, 1982), the direct product space is made up of above two spaces. Every operators and state in the Hilbert space has corresponding operators and state in the tilde space. A and A^+ have corresponding operators $\tilde{A}$ and $\tilde{A}^+$ which obey

$$\left[\tilde{A}, \tilde{A}^+\right] = 1, \left[\tilde{A}, A\right] = \left[\tilde{A}, A^+\right] = \left[A, \tilde{A}^+\right] = \left[A^+, \tilde{A}^+\right] = 0 \tag{13}$$

Thermal vacuum state $|0\tilde{0}\rangle$ is zero temperature in H–T direct product space, thermal vacuum state can be obtained by thermal canonical Bogoliubov commutation expressing with $T(\theta)$. We get

$$\left|0\tilde{0}\right\rangle_T = T(\theta)\left|0\tilde{0}\right\rangle \tag{14}$$

where $|0\tilde{0}\rangle_T$ stands for thermal vacuum state. $T(\theta) = \mathrm{e}^{-\theta(A\tilde{A}-A^+\tilde{A}^+)}$, the θ relates with count average thermal particle amount n_0, $n_0 = \sinh^2\theta$, we obtain relation of n_0 and temperature by Boson–Einstein distribution

$$n_0 = \frac{1}{\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1}, \tag{15}$$

where k_B is Boltzmann constant. Thermal destruction and creation operator $A(\theta)$, $A^+(\theta)$ are

$$A(\theta) = T(\theta)AT^+(\theta) = \cosh\theta A - \sinh\theta\tilde{A}^+, \tag{16a}$$

$$A^+(\theta) = T(\theta)A^+T^+(\theta) = \cosh\theta A^+ - \sinh\theta\tilde{A}. \tag{16b}$$

Where we have used relation $\mathrm{e}^B A\mathrm{e}^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \frac{1}{3!}[B, [B, [B, A]]] + \cdots$. From Eqs. (10) and (16), we obtain easily

$$I(\theta) = \sqrt{\frac{\hbar}{2L\omega}}\left[\cosh\theta(A + A^+) - \sinh\theta\left(\tilde{A} + \tilde{A}^+\right)\right], \tag{17a}$$

$$U(\theta) = \frac{1}{j}\sqrt{\frac{L\omega\hbar}{2}}\left[\cosh\theta(A - A^+) + \sinh\theta\left(\tilde{A} - \tilde{A}^+\right)\right]. \tag{17b}$$

Therefore, we have

$${}_T\left\langle 0\tilde{0}\right| I(\theta)\left|0\tilde{0}\right\rangle_T = 0, \tag{18a}$$

$${}_T\left\langle 0\tilde{0}\right| I^2(\theta)\left|0\tilde{0}\right\rangle_T = \frac{\hbar}{2L\omega}(2n_0 + 1), \tag{18b}$$

$${}_T\left\langle 0\tilde{0}\right| U(\theta)\left|0\tilde{0}\right\rangle_T = 0, \tag{18c}$$

$$_T\left\langle 0\tilde{0}\right| U^2(\theta) \left|0\tilde{0}\right\rangle_T = \frac{L\omega\hbar}{2}(2n_0+1). \tag{18d}$$

4. THE QUANTUM FLUCTUATIONS OF MESOSCOPIC QUARTZ PIEZOELECTRIC CRYSTAL IN THE THERMAL VACUUM STATES

When electrical source is switched off, $i_S(t) = 0$, the quantum fluctuations of the voltage and current of equivalent circuit each loop in the thermal vacuum states $|0\tilde{0}\rangle_T$ are researched.

4.1. The Quantum Fluctuations of the Voltage and Current of Inductance *L*

From Eqs. (5), (18), (11), and (13), we obtain

$$_T\langle i\rangle_T = 0, \quad _T\langle i^2\rangle_T = \frac{\hbar}{2L\omega}\tau^{-2}\mathrm{e}^{-\frac{R}{L}t}(2n_0+1); \tag{19a}$$

$$_T\langle u\rangle_T = 0, \quad _T\langle u^2\rangle_T = \frac{L\hbar}{2\omega}\omega_0^2\tau^{-2}\mathrm{e}^{-\frac{R}{L}t}(2n_0+1). \tag{19b}$$

Therefore, the quantum fluctuations of the voltage and current of inductance L are

$$_T\left\langle(\Delta i)^2\right\rangle_T = \frac{\hbar}{2L\omega}\tau^{-2}\mathrm{e}^{-\frac{R}{L}t}(2n_0+1), \tag{20a}$$

$$_T\left\langle(\Delta u)^2\right\rangle_T = \frac{L\hbar}{2\omega}\omega_0^2\tau^{-2}\mathrm{e}^{-\frac{R}{L}t}(2n_0+1). \tag{20b}$$

Substituting Eq. (15) into Eq. (20), we obtain

$$_T\langle(\Delta i)^2\rangle_T = \frac{\hbar}{2L\omega}\tau^{-2}\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_BT}\right), \tag{21a}$$

$$_T\langle(\Delta u)^2\rangle_T = \frac{L\hbar}{2\omega}\omega_0^2\tau^{-2}\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{21b}$$

So the uncertainty relation is

$$_T\langle(\Delta i)^2\rangle_T \cdot {}_T\langle(\Delta u)^2\rangle_T = \frac{\hbar^2}{4}\frac{1}{1-R^2C_0C/[4L(C_0+C)]}\tau^{-4}\,\mathrm{e}^{-\frac{2R}{L}t}\coth^2\left(\frac{\hbar\omega}{2k_BT}\right). \tag{22}$$

When $T \to 0$, $\coth\left(\frac{\hbar\omega}{2k_BT}\right) \to 1$, we obtain

$$\langle(\Delta i)^2\rangle = \frac{\hbar}{2L\omega}\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}, \tag{23a}$$

$$\langle(\Delta u)^2\rangle = \frac{L\hbar}{2\omega}\omega_0^2\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}. \tag{23b}$$

$$\langle(\Delta i)^2\rangle\langle(\Delta u)^2\rangle = \frac{\hbar^2}{4}\frac{1}{1 - R^2C_0C/[4L(C_0 + C)]}\tau^{-4}\,\mathrm{e}^{-\frac{2R}{L}t}. \tag{24}$$

which reduces to the results of Li (2005).

4.2. The Quantum Fluctuations of the Voltage and Current of Ullage Resistance *R*

From Fig. 1, we have $u_R = iR, \quad i_R = i$. From Eq. (21), we obtain easily

$$_T\langle(\Delta i)^2\rangle_T = \frac{\hbar}{2L\omega}\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right), \tag{25a}$$

$$_T\langle(\Delta u)^2\rangle_T = \frac{R^2\hbar}{2L\omega}\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right). \tag{25b}$$

4.3. The Quantum Fluctuations of the Voltage and Current Transformation Relating with Time in Series Capacitance *C*

From Fig. 1, we have $u_R = iR,\ i_R = i$. Therefore, we obtain

$$_T\langle(\Delta i_C)^2\rangle_T = \frac{\hbar}{2L\omega}\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right). \tag{26}$$

From $i_C = \frac{C\,du_C}{dt}$ and $i_C = i$, we obtain

$$_T\langle(\Delta \dot{u}_C)^2\rangle_T = \frac{\hbar}{2C^2L\omega}\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right). \tag{27}$$

4.4. The Quantum Fluctuations of the Voltage and Current Transformation Relating with Time in Parallel Connection Capacitance C_0

If taking $i_\mathrm{S}(t) = 0$, from Fig. 1, we have $i_{C0} = -i.$, one has

$$\dot{u}_{C0} = -\frac{i}{C_0}, \tag{28}$$

therefore, the quantum fluctuations of the voltage and current transformation relating with time in parallel connection capacitance C_0 are

$$_T\langle(\Delta i_{C0})^2\rangle_T = \frac{\hbar}{2L\omega}\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right), \tag{29}$$

$$_T\langle(\Delta \dot{u}_{C0})^2\rangle_T = \frac{\hbar}{2C_0^2L\omega}\tau^{-2}\,\mathrm{e}^{-\frac{R}{L}t}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right). \tag{30}$$

5. CONCLUSIONS

Depending on the classical equations of motion, the mesoscopic quartz piezoelectric crystal equivalent circuit is quantized by the method of damped harmonic oscillator quantization. The quantum fluctuations of the voltage and current in mesoscopic quartz piezoelectric crystal equivalent circuit each loop in the thermal vacuum states are researched. The results show that (i) the quartz piezoelectric crystal equivalent circuit is equivalent with quantum harmonic oscillator in the mesoscopic conditions, (ii) the quantum fluctuations of the voltage and current of each loop relating with not only the parameter of the equivalent circuit element but also the temperature in the thermal vacuum states, the higher the temperature is, the more quantum fluctuations exhibits, the lower the temperature is, the smaller quantum fluctuations exhibits, (iii) because of ullage resistance R, the quantum fluctuations of the voltage and current decay along with time.

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